The Analogy between Inductively-defined Sets and Computably Enumerable Sets

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There is an analogy between inductively-defined sets and computably enumerable sets that seems to have been observed many times before, which gives an analogy between inductive-coinductive duality and the duality between c.e. and co-c.e. sets.

The predicative construction of an inductive set begins with the empty set and applies the associated operator iteratively *enumerating* the elements of the least fixed point. If further, this construction is in some sense computable, then this should give a computable enumeration of the inductive set.

Dually, the predicative construction of an coinductive set begins with the entire set and iteratively removes elements just how non-membership in a co-c.e. set is computed.

Old and well-known results in the literature on inductive definitions give a way to make this analogy formal. All results presented here can be found in Hinman's book *Recursion-theoretic Hierarchies*.

Theorem 1.1. *The least fixed point of a computable monotone operator is computably enumerable. Moreover, every computably enumerable set is 1-1 reducible to the least fixed point of a computable monotone operator.*

If we broaden the class of monotone operators to Σ^0_1 operators, then we get something slightly tighter.

Theorem 1.2. *The least fixed point of a Σ^0_1 computable monotone operator is computably enumerable. Moreover, every computably enumerable set is the least fixed point of some Σ^0_1 monotone operator.*

The analogy can also be made formal by taking the perspective of inductively-defined sets as being generated by rules and enumeration operators. Below I try to give intuitive sketches for some of the statements.

Definition 1.3 (stages)**.** Let $F : P(X) \to P(X)$ be a monotone operator. By transfinite recursion, we define

$$F^{\alpha} := F(F^{<\alpha})$$

with

$$F^{<\alpha} := \bigcup_{\xi<\alpha} F^{\xi}.$$

Then $\bar{F} := \bigcup_{\alpha \in \mathsf{Ord}} F^{\alpha}$.

It's a basic result of fixed point theory that $\bar{F} = \mu F$ and that there is a least ordinal λ such that $F^{<\lambda} = F^{\lambda}$ where $|\lambda| \leq |X|$.

Definition 1.4. The **closure ordinal** $|F|$ of F is the least ordinal λ such that

$$F^{<\lambda} = F^{\lambda}.$$

Definition 1.5.

1. An operator $F : \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ is **computable**[1] if there is a total Turing functional $\Phi : \mathscr{P}(\omega) \to 2$ such that

$$x \in F(A) \implies \Phi^A(x) = 1$$
$$x \notin F(A) \implies \Phi^A(x) = 0$$

2. An operator $F : \mathscr{P}(\omega) \to \mathscr{P}(\omega)$ is Σ^0_1 if there is a partial Turing functional $\Phi : \mathscr{P}(\omega) \to \mathscr{P}(\omega)$

$$x \in F(A) \iff \Phi^A(x) \downarrow= 1.$$

Proposition 1.6. *If F is a computable monotone operator, then*

1. *the sequence*

$$(F^n)_{n\in\omega}$$

   is uniformly computable,

2. $|F| \leq \omega$, *and*

3. $\overline{F}$ *is* Σ^0_1.

The essential idea is that use is finite and any element in the ωth stage of the construction was already added at some finite stage.

Proof.

1. Let Φ witness that F is computable. Given inputs n, x, we want a uniform to decide $x \in F^n$. If $n = 0$, we just run the total procedure $\Phi^{\varnothing}(x)$, and if $n = k + 1$, then we run $\Phi^{F^k}(x)$ and queries $\mathrm{y} \in \mathrm{F}^{\mathrm{k}}$ are computed recursively.

2. Let $x \in F^\omega = F(F^{<\omega})$, then $\Phi^{F^{<\omega}}(x) = 1$. Since the use of this computation is finite and the sets $(F^n)_{n<\omega}$ are increasing, then there is a maximal m such that $\Phi^{F^m}(x) = 1$. Thus, $x \in F^{<\omega}$.

3. Since the closure ordinal is $\leq \omega$, it suffices to computably enumerate the union $\bigcup_{n<\omega} F^n$, but since this is a uniformly computable sequence, then this is immediate.

□

In this context, what is computable about the iterative construction is that we can computably decide if some number x is in the nth stage of the construction of the inductive set.

Lemma 1.7. *Let F be a* Σ^0_1 *monotone operator, then* $|F| \leq \omega$.

Lemma 1.8. *Let F be a* Σ^0_1 *monotone operator, then there is a computable relation R such that for all* $x \in \omega$ *and* $A \in 2^\omega$,

$$x \in F(A) \iff \exists n[D_n \subseteq A \wedge R(x, n)].$$

Proposition 1.9. *Let F be a* Σ^0_1 *operator, then* $\overline{F}$ *is computably enumerable.*

By the duality for monotone operators, we get the dual result for Π^0_1 operators and co-c.e. sets.

[1]More generally, the complexity of an operator is the complexity of the relation $x \in F(A)$ as a subset of $\omega \times 2^\omega$.